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Abstract: In the standard viewpoint, the temperature of a stationary black hole is proportional to its surface gravity. This is a semiclassical result and the quantum gravity effects are not taken into consideration. This research explores a unified expression for the black hole temperatures in the sense of a generalized uncertainty principle (GUP). Our argument is based on a heuristic analysis of a particle which is absorbed by the black hole. We make a new proposal that the square root of the black hole area represents the characteristic size in the absorption process (i.e. $\rho_0 \sim \sqrt{A}$), which is valid to a class of static and spherically symmetric black holes and a Kerr-Newman case. The information capacity of a remnant is also discussed by Bousso's D-bound in de Sitter spacetime.

Contents

1 Introduction

Heisenberg's uncertainty relation is one of the fundamental principles of quantum mechanics. This principle only involves the quantum effects of matters, and it does not directly describe the quantum fluctuations of spacetimes. However, many efforts have shown that Heisenberg's principle may suffer a modification [1]–[13], in the context of quantum gravity. Concretely, a generalized uncertainty principle(GUP) reads

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha}{\hbar}\Delta p, \tag{1.1}$$

where $\alpha \sim G$. The second term on the r.h.s means a new duality, which is historically related to the scattering amplitude of high energy string [1, 3] and the spacetime uncertainty principle [2, 4]. This term is also attributed to gravity in some gedanken experiments [7, 8, 10, 12]. Different from Heisenberg's principle, GUP restricts the shortest distance that we can probe (i.e. $\Delta x \geq 2\sqrt{\alpha} \sim l_p$). This agrees with the belief that Planck length is a fundamental scale in quantum gravity.

Since uncertainty principle is of great importance to quantum physics, GUP has caused extensive interests and arguments. In particular, GUP's effects on the thermodynamics of a Schwarzschild black hole have been discussed by a heuristic method [13]. The crucial idea therein is that Δx and Δp are identified as the black hole's size and temperature respectively. An interesting result is that the black hole mass is not allowed to be less than a scale of order Planck mass, which suggests a black hole remnant. Although GUP's impacts on black hole thermodynamics have been discussed in the literature [13]–[28], a universal expression is still absent.

In the semiclassical framework, Hawking temperature of a stationary black hole is proportional to the surface gravity, i.e.

$$T_H = \frac{\hbar\kappa}{2\pi}, \tag{1.2}$$

where Planck constant reveals the quantum nature of black hole radiation. In the Bekenstein's original work [30], Heisenberg's uncertainty principle is crucial to the linear relation between Hawking temperature and surface gravity.[1] In our opinion, GUP changes the semiclassical framework to a certain context, and the semiclassical black hole temperature (1.2) should suffer a modification. How the expression (1.2) is corrected by GUP? This research explores a temperature expression in the sense of GUP, which is expected to be valid for more general black holes besides Schwarzschild case. We discuss a class of static and spherically symmetric black holes, as well as a Kerr-Newman black hole. The temperatures of these black holes have the same form. The information capacity of a black hole remnant is discussed in de Sitter spacetime, in terms of a Bousso's D-bound. Enlightened by refs. [16, 29], we follow the Bekenstein's original work [30], and analyze a gedanken experiment that a neutral particle just outside the horizon is absorbed by the black hole. This research takes the units $G = c = k_B = 1$.

2 Black hole thermodynamics

2.1 General consideration

This subsection makes a note of the basis for further discussion. Let us start with the first law of black hole mechanics [30, 34]

$$dM = \frac{\kappa}{8\pi}dA + \sum_i Y_i dy_i, \tag{2.1}$$

where the terms $\sum_i Y_i dy_i$ represent the work done on the black hole by an external agent. y_i represents one of the black hole's variables such as electronic charge or angular momentum; Y_i is the generalized force corresponding to the variable y_i, e.g. electrostatic potential or angular velocity. The above formula is a result of classical general relativity. However, it has been endowed with thermodynamic meaning since Hawking radiation was discovered, i.e.

$$dM = TdS + \sum_i Y_i dy_i. \tag{2.2}$$

Corresponding to the standard temperature (1.2), the black hole entropy is expressed as $S_{BH} = (4\hbar)^{-1}A$, i.e. the so-called Bekenstein-Hawking entropy. However, this simple relation is a semiclassical result. In more general situations, the entropy of a black hole is assumed to be a function of its area [30], $S = S(A)$. Following from (2.1) and the definition of thermodynamics, the temperature is expressed as

$$T = \left(\frac{\partial M}{\partial S}\right)_{y_i} = \frac{dA}{dS} \times \left(\frac{\partial M}{\partial A}\right)_{y_i} = \frac{dA}{dS} \times \frac{\kappa}{8\pi}, \tag{2.3}$$

where the variables y_i are fixed. The temperature expression is determined by the relation between the entropy and area. In order to find the concrete form of $S(A)$, we consider

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[1]This point is also stressed in ref. [33], where the linear relation $T_H \sim \hbar\kappa$ can be obtained by another heuristic method via Heisenberg's uncertainty principle.

a particle captured by the black hole. When the particle disappears, on one hand, its information is lost to an observer outside the horizon; on the other hand, the smallest increase in the area of a Kerr-Newmann black hole is given by [30]

$$\Delta A \sim b\mu, \tag{2.4}$$

where b and μ are the particle's size and mass, respectively. Identifying the loss of information with the increase of black hole entropy, we obtain

$$\Delta S \simeq \frac{dS}{dA}\Delta A.$$

According to information theory, the loss of information is one bit at least, i.e. $(\Delta S)_{min} = \ln 2$. The next step is to consider the physical limitations on μ and b, since they are crucial for ΔA to be minimized. For a classical particle (point-like object), $(\Delta A)_{\min} = 0$. However, in quantum mechanics, a particle is described by a wave packet and a definite trajectory does not exist. The width of wave packet is defined as the standard deviation of x distribution (i.e. the position uncertainty), which can be interpreted as the characteristic size of the particle ($b \sim \Delta x$). Furthermore, the momentum uncertainty is not allowed to be greater than the mass ($\Delta p \leq \mu$), in the process of measuring the particle's position. Otherwise the relativistic effects lead to the creation of a partner of the particle and make the measurement meaningless. Thus the expression (2.4) is deduced to

$$\Delta A \sim b\mu \geq \Delta x \Delta p. \tag{2.5}$$

The smallest increase in area cannot be arbitrarily small and it is restricted by the uncertainty relation of quantum mechanics. In the Bekenstein's insightful work, Heisenberg principle is utilized to identify the particle's size with the Compton wavelength of itself, and then the minimum increase in horizon area is given by $\Delta A \sim l_p^2$. This results in

$$\frac{\Delta S}{\Delta A} = \text{const},$$

which means the linear relation between the black hole entropy and the horizon area. GUP will correct the Bekenstein's result. Substituting (1.1) into (2.5), we have

$$\Delta A \geq \gamma_1 \hbar \left[1 + \frac{\alpha}{\hbar^2}(\Delta p)^2 \right], \tag{2.6}$$

where γ_1 is a calibration factor. In order for ΔA to be minimized, we should take the smallest uncertainty of momentum. Following from (2.6), the minimum increase in area, $(\Delta A)_{\min}$, would be a constant if $\Delta p \to 0$. At a first glimpse, there seems to be no correction to the Bekenstein's result. However, $\Delta p \to 0$ means $\Delta x \to \infty$. For a particle captured by black hole, Δp is not allowed to be arbitrarily small, since the particle is confined within a finite region and Δx is finite. $(\Delta A)_{\min}$ is therefore no longer a constant, which results in some corrections to the linear relation between entropy and area. In the following subsections, a static and spherically symmetric black hole as well as an axially symmetric Kerr-Newman black hole are discussed respectively.

2.2 A class of static and spherically black holes

We consider a static and spherical black hole as follows

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the horizon is located by $F(r_0) = 0$. The above line element describes a class of static and spherically symmetric black holes, such as Schwarzschild, Reissner-Nordström and their partners in (anti-)de Sitter spacetime. When a particle is captured by black hole, Δx should not be greater than a specific scale which minimizes ΔA. This characteristic size should be related to the black hole, if $(\Delta A)_{\min}$ is expected to reflect an intrinsic property of the horizon. For a static and spherically symmetric black hole, it is identified with the twice radius of horizon,[2] i.e.

$$2r_0 \geq \Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha}{\hbar}\Delta p, \tag{2.7}$$

which imposes a constraint on the momentum uncertainty as follows

$$\frac{\hbar}{\alpha}\left[r_0 - \sqrt{r_0^2 - \alpha}\right] \leq \Delta p \leq \frac{\hbar}{\alpha}\left[r_0 + \sqrt{r_0^2 - \alpha}\right]. \tag{2.8}$$

So the product of Δx and Δp yields

$$\begin{aligned} \Delta x \Delta p &\geq \hbar\left[1 + \frac{\alpha}{\hbar^2}(\Delta p)^2\right] \\ &\geq \frac{2\hbar}{\alpha}\left(r_0^2 - r_0\sqrt{r_0^2 - \alpha}\right) = \hbar', \end{aligned} \tag{2.9}$$

where the second inequality is obtained by taking the lower bound of Δp. The above inequality can be rewritten as a Heisenberg-type uncertainty principle, $\Delta x \Delta p \geq \hbar'$, where $\hbar'$ may be regarded as an effective Planck constant. Thus the increase in area satisfies

$$\Delta A \geq \gamma_1 \hbar' = \frac{2\gamma_1 \hbar}{\alpha}\left(r_0^2 - r_0\sqrt{r_0^2 - \alpha}\right). \tag{2.10}$$

When the particle vanishes, the information of one bit is lost and the black hole acquires the increase in entropy $(\Delta S)_{\min} = \ln 2$. On the other hand, the minimum increase in the horizon area is given by the lower bound of (2.10), which is denoted by $(\Delta A)_{\min}$. We obtain

$$\frac{dA}{dS} \simeq \frac{(\Delta A)_{\min}}{(\Delta S)_{\min}} = \frac{2\gamma_1 \hbar}{\alpha \ln 2}\left(r_0^2 - r_0\sqrt{r_0^2 - \alpha}\right). \tag{2.11}$$

The black hole temperature (2.3) is deduced to

$$T \simeq \frac{\kappa}{8\pi} \cdot \frac{2\gamma_1 \hbar}{\alpha \ln 2}\left(r_0^2 - r_0\sqrt{r_0^2 - \alpha}\right),$$

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[2]For example, see ref. [13]. This identification is compatible with the suggestion that $(\Delta x)_{\max}$ should be represented by the irreducible mass(for a rotating black hole), which is based on the consideration that the entropy is unchanged in a reversible process(see the discussion in the next subsection).

which is not only related to the surface gravity but also to the black hole size. It should reproduce the standard result $T = \kappa/2\pi$, as $\alpha \to 0$. This requires that the calibration factor yield $\gamma_1 = 4\ln 2$. Thus we obtain

$$T \simeq \frac{\hbar'\kappa}{2\pi}, \tag{2.12}$$

which is the expression for the temperature of a static and spherically symmetric black hole. Comparing the standard formula (1.2) with the revised version (2.12), we find that the latter can be obtained from the former by substituting $\hbar'$ for the Planck constant. It suggests that $\hbar'$ play the role of an effective Planck constant.

The expression (2.12) can be understood by reexamining the efficiency of a Geroch process. This gedanken experiment imagines a machine operating between a black hole and a remote reservoir.[3] In this process, a box is filled with black body radiation from the reservoir and lowered down to the black hole surface. After emitting the radiation into the black hole, the box is moved away from the black hole. The over-all process converts heat into work with the efficiency [30]

$$\eta = 1 - \gamma_2 \kappa \ell, \tag{2.13}$$

where ℓ is the size of the box, and γ_2 is a coefficient factor to be determined. The smaller ℓ is, the greater η is. In practical situations, it is reasonable that the box's size is required to yield $\ell \leq 2r_0$. This is also necessary to emit the total radiation into the black hole, otherwise the photons with lower energy will not contribute to the Geroch process. On the other hand, the box must be big enough for the wavelengths of the radiation, and ℓ has a minimum value which is related to the temperature of radiation, T_R. To find the relation between the temperature and efficiency, we rewrite (2.13) as

$$\eta = 1 - \gamma_2(\ell T_R)\frac{\kappa}{T_R}. \tag{2.14}$$

As the mean energy of thermal photons, the radiation temperature yields $T_R > \epsilon$, where ϵ is the photon's minimum energy which is given by the lower bound of (2.8). Thus we obtain

$$\begin{aligned}
\ell T_R > \epsilon\ell &= \frac{\hbar}{2\alpha}\left(\ell^2 - \ell\sqrt{\ell^2 - 4\alpha}\right) \\
&\geq \frac{2\hbar}{\alpha}\left(r_0^2 - r_0\sqrt{r_0^2 - \alpha}\right) \\
&= \hbar',
\end{aligned}$$

where we have considered $\ell \leq 2r_0$. Thus the efficiency (2.14) yields

$$\eta < 1 - \gamma_2\frac{\hbar'\kappa}{T_R}. \tag{2.15}$$

Comparing it with the efficiency of a heat engine operating between two reservoirs, we find that the expression $\hbar'\kappa$ plays the role of the black hole temperature. This agrees with (2.12), up to a constant factor.

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[3]For details, see ref. [30].

The black hole entropy is given by

$$S = \int \frac{dS}{dA} dA \simeq \int \frac{(\Delta S)_{\min}}{(\Delta A)_{\min}} dA.$$

Considering (2.11) and setting $\gamma_1 = 4 \ln 2$, we obtain

$$\begin{aligned}
S &\simeq \frac{1}{4} \int \frac{dA}{\hbar'} \\
&= \frac{\pi}{\hbar} \int \left(r_0 + \sqrt{r_0^2 - \alpha} \right) dr_0 \\
&= \frac{\pi}{2\hbar} \left[r_0^2 + r_0 \sqrt{r_0^2 - \alpha} - \alpha \ln \left(r_0 + \sqrt{r_0^2 - \alpha} \right) \right].
\end{aligned} \tag{2.16}$$

When $r_0 \gg \sqrt{\alpha}$, Bekenstein-Hawking entropy and the log-area correction are presented as

$$S = (4\hbar)^{-1}(A - \alpha\pi \ln A + \cdots). \tag{2.17}$$

In the context of the GUP, the heat capacity is given by

$$\begin{aligned}
C &= T \frac{\partial S}{\partial T} = \frac{\hbar' \kappa}{2\pi} \cdot \frac{\partial S}{\partial A} \cdot \frac{\partial A}{\partial T} \\
&= \frac{1}{4} \left(\frac{\partial \hbar'}{\partial A} + \hbar' \kappa^{-1} \frac{\partial \kappa}{\partial A} \right)^{-1}.
\end{aligned} \tag{2.18}$$

Direct calculation gives

$$\frac{\partial \hbar'}{\partial A} = \frac{1}{8\pi r_0} \frac{\partial \hbar'}{\partial r_0} = -\frac{\Delta \hbar}{4f}.$$

where $\Delta \hbar = \hbar' - \hbar$, $f = f(r_0) = \pi r_0 \sqrt{r_0^2 - \alpha}$. The heat capacity (2.18) is deduced to

$$C = C_0 f \left[\frac{\hbar'}{\hbar} f - C_0 \Delta \hbar \right]^{-1}, \tag{2.19}$$

where

$$C_0 = T_H \frac{\partial S_{\mathrm{BH}}}{\partial T_H} = (4\hbar)^{-1} \kappa \frac{\partial A}{\partial \kappa},$$

is the standard heat capacity defined by the Hawking temperature (1.2) and Bekenstein-Hawking entropy.

In the derivation of (2.12), we assume that the black holes in an asymptotically de Sitter spacetime also yield the law (2.4). As an example, we consider a Reissner-Nordström-de Sitter black hole whose horizon radius r_0 is determined by

$$0 = F(r_0) = 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} - \frac{\Lambda}{3} r_0^2.$$

The first law is similar to eq. (2.1), where the surface gravity is [31]

$$\begin{aligned}
\kappa = \frac{F'(r_0)}{2} &= r_0^{-1} \left(\frac{M}{r_0} - \frac{Q^2}{r_0^2} - \frac{\Lambda}{3} r_0^2 \right) \\
&= (2r_0)^{-1} \left[1 - \Lambda r_0^2 - \frac{Q^2}{r_0^2} \right].
\end{aligned} \tag{2.20}$$

When the black hole captures a neutral particle, the first law becomes

$$dM = \frac{\kappa}{8\pi} dA. \qquad (2.21)$$

On the other hand, based on a Bekenstein-type analysis, the smallest increase in the black hole mass is given by [32]

$$\Delta M \sim b\mu\kappa, \qquad (2.22)$$

where b and μ are the particle's size and mass respectively. Considering (2.21) and (2.22), the smallest increase in horizon area is $\Delta A \sim b\mu$, which is just (2.4).

Let us give a remark on the difference from ref. [13]. In ref. [13], Δx and Δp are identified with the black hole's radius and temperature respectively. This leads to a deduction that the temperature becomes an explicit function only of the black hole size, and cannot reproduce eq. (1.2) as $\alpha \to 0$. A possible extension of ref. [13] is to identify Δx with the inverse surface gravity [16], and then the GUP (1.1) would give the temperature as follows

$$T' = \frac{\hbar\kappa}{\pi \left(1 + \sqrt{1 - \alpha\kappa^2/\pi^2}\right)}, \qquad (2.23)$$

which can reproduce the standard result (as $\alpha \to 0$). Obviously, the above expression is different from eq. (2.12). The entropy would be

$$\begin{aligned}
S' &= \int \frac{dM - Y_i dy_i}{T} = \int \frac{\kappa}{8\pi T} dA \\
&= \frac{1}{8\hbar} \int \left(1 + \sqrt{1 - \alpha\kappa^2/\pi^2}\right) dA \\
&= \frac{A}{4\hbar} - \frac{\alpha}{16\pi^2\hbar} \int \kappa^2 dA + \cdots.
\end{aligned} \qquad (2.24)$$

The higher order terms of α can be ignored, when a big black hole is considered($\alpha\kappa^2 \ll 1$). Considering (2.20), the entropy (2.24) becomes

$$S' = \frac{A}{4\hbar}\left(1 + \frac{\alpha\Lambda}{8\pi^2}\right) - \frac{\alpha}{16\pi\hbar}(1 + 2Q^2\Lambda)\ln A - \frac{\alpha\Lambda^2}{2\hbar \times (16\pi)^3}A^2 - \frac{Q^2\alpha}{2\hbar A} + \frac{\pi Q^4\alpha}{2\hbar A^2}. \quad (2.25)$$

Besides the correction to the prefactor of Bekenstein-Hawking entropy, there is a term of A^2. We also notice that the charge Q appears in (2.25). Generally, (2.24) is an explicit function of A and other variables(e.g. charge and angular momentum), since κ cannot be expressed as a function only of A. These features are distinct from the entropy expression (2.17).

2.3 Kerr-Newman black hole

For a Kerr-Newman black hole of mass M, charge Q and angular momentum $J = aM$, the first law of mechanics is [30]

$$dM = \frac{\kappa}{8\pi} dA + \phi dQ + \Omega dJ, \qquad (2.26)$$

– 7 –

where $A = 4\pi(r_+^2 + a^2)$ is the area of the event horizon. In Boyer-Lindquist coordinates [36], the location of the horizon is given by

$$r_+ = M + \sqrt{M^2 - Q^2 - a^2}. \tag{2.27}$$

Here we don't present the expressions for the surface gravity κ, electric potential ϕ, and angular velocity Ω, since they are unimportant for the following discussion.[4]

In the previous subsection, we suggest that for a particle captured by the black hole, the position uncertainty Δx yields

$$\Delta x \leq 2\rho_0, \tag{2.28}$$

For a static and spherically symmetric black hole, the characteristic size ρ_0 is identified with the twice radius of the horizon. We are confronted with a question of understanding the meaning of ρ_0, when a rotating black hole is considered. At a first glimpse, it appears natural that ρ_0 is represented by r_+ [35]. However, this proposal is doubtable, although it is workable for the static and spherically symmetric cases. This is because Boyer-Lindquist coordinates are different from ordinary polar coordinates. For instance, in rectangular coordinates (X, Y, Z), $r = const$ represents an ellipsoid rather than a sphere. Concretely speaking, the coordinates (r, θ, ϕ) are related to the rectangular coordinates by [36–38]

$$
\begin{aligned}
X &= \sqrt{r^2 + a^2} \sin\theta \cos\varphi^*, \\
Y &= \sqrt{r^2 + a^2} \sin\theta \sin\varphi^*, \\
Z &= r \cos\theta,
\end{aligned}
\tag{2.29}
$$

where

$$\varphi^* = \varphi - \tan^{-1}\frac{a}{r} - a \int_\infty^r \frac{dr}{\Delta}.$$

Following from (2.29), we obtain

$$\frac{X^2 + Y^2}{r^2 + a^2} + \frac{Z^2}{r^2} = 1, \tag{2.30}$$

which is axially symmetric. Obviously, the surface of a Kerr-Newman black hole$(r \to r_+)$ is a confocal ellipsoid. There are two characteristic sizes: r_+ and $\sqrt{r_+^2 + a^2}$. Which is ρ_0? In order to minimize ΔA, we choose the latter, i.e.

$$\rho_0 = \sqrt{r_+^2 + a^2}. \tag{2.31}$$

One of the evidences for (2.31) is that the absorption cross section for a Kerr-Newman black hole is proportional to its area [39], $\sigma_{abs} \sim A = 4\pi\rho_0^2$, which can be interpreted by the aid of a two-body process in an effective string theory that describes the collective excitations of the black hole at weak coupling [40–42]. This means that ρ_0 is indeed a characteristic size in the absorption process.

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[4]The expressions for these quantities were presented in the literature, e.g. ref. [30] and some text books.

Furthermore, as argued immediately, (2.31) is a reasonable choice in the sense of thermodynamics, which can be explained along another line of arguments. Let us return to (2.28), where ρ_0 is to be determined. Replacing r_0 with ρ_0 and repeating the procedure from (2.7) to (2.11), we obtain

$$\frac{(\Delta A)_{\min}}{(\Delta S)_{\min}} = \frac{2\gamma_1\hbar}{\alpha\ln 2}\left(\rho_0^2 - \rho_0\sqrt{\rho_0^2 - \alpha}\right). \tag{2.32}$$

If ρ_0 is identified directly with r_+, (2.32) becomes

$$\begin{aligned}
\frac{(\Delta A)_{\min}}{(\Delta S)_{\min}} &= \frac{2\gamma_1\hbar}{\alpha\ln 2}\left(r_+^2 - r_+\sqrt{r_+^2 - \alpha}\right) \\
&= \frac{\gamma_1\hbar}{2\pi\alpha\ln 2}\left[A - 4\pi a^2 - \sqrt{(A - 4\pi a^2 - 2\alpha\pi)^2 - 4\alpha^2\pi^2}\right],
\end{aligned}$$

which means that the entropy depends on two quantities: A and a. This contradicts Bekenstein's assumption that the entropy of a black hole is a function only of its area [30], and would lead to a deduction incompatible with thermodynamics. Supposing $S = S(A, a)$, we have

$$\begin{aligned}
dS &= \frac{\partial S}{\partial A}dA + \frac{\partial S}{\partial a}da \\
&= \frac{\partial S}{\partial A}dA + M^{-1}\frac{\partial S}{\partial a}(dJ - a\,dM). \tag{2.33}
\end{aligned}$$

In a reversible process, the black hole area is unchanged [43, 44], $dA = 0$. So the change in black hole mass is attributed to the work done by an external agent which changes the black hole's charge and angular momentum, and the first law (2.26) becomes

$$dM = \phi\,dQ + \Omega\,dJ.$$

eq. (2.33) is therefore rewritten as

$$dS = M^{-1}\frac{\partial S}{\partial a}\left[(1 - a\Omega)dJ - a\phi\,dQ\right], \tag{2.34}$$

which means $dS \neq 0$ if $(\partial S/\partial a)_A \neq 0$, since Q and J are independent variables. Especially for a neutral black hole, we have

$$dS = M^{-1}\frac{r_+^2}{r_+^2 + a^2}\frac{\partial S}{\partial a}dJ.$$

The black hole entropy could be changed by an external agent which changes the angular momentum reversibly, if $(\partial S/\partial a)_A \neq 0$. This means that the entropy is not invariant in such a reversible process. This contradicts with the basic concept of thermodynamics. The crucial reason is that ρ_0 is improperly interpreted as r_+.

What is ρ_0? Enlightened by the above discussion, ρ_0 should be unchanged in a reversible process. This is required by the fact that S and A are invariant in the same process. Following from (2.32), the black hole entropy is expressed as $S = S(A, \rho_0)$, so we have

$$dS = \frac{\partial S}{\partial A}dA + \frac{\partial S}{\partial \rho_0}d\rho_0.$$

$d\rho_0 = 0$ as $dS = dA = 0$, namely, ρ_0 is an invariant in a reversible process. For a rotating black hole in a reversible process, its irreducible mass M_{ir} is unchanged [43, 44], where

$$M_{ir} = \sqrt{\frac{A}{16\pi}} = \frac{1}{2}\sqrt{r_+^2 + a^2}. \tag{2.35}$$

This similarity implies that ρ_0 should be interpreted as the black hole irreducible mass, $\rho_0 \sim M_{ir}$. This can be understood in another manner. We notice that (2.32) involves three quantities of a black hole: the area A, entropy S and the characteristic size ρ_0. ρ_0 is thus related not only to A but also to S. In other words, ρ_0 is a bridge which crosses the gap between A and S, hence it must have geometric and thermodynamic meanings. The black hole irreducible mass agrees with this requirement. On one hand, M_{ir} is related to the horizon area by (2.35). On the other hand, M_{ir} is the energy that can not be extracted by a classical process (e.g. Penrose process). In the thermodynamic sense, M_{ir} corresponds to the degraded energy that can not be transformed into work [30]. As a measure for the degradation of energy, the entropy is related to the irreducible mass by $S = S(M_{ir})$. Thus ρ_0 is endued with geometric and thermodynamic meanings by identifying it with M_{ir}. Therefore, (2.31) is a natural choice in the context of thermodynamics.

The next thing to be done is similar to the previous subsection. Replacing r_0 with ρ_0, we obtain the temperature, entropy and heat capacity of a Kerr-Newman black hole, i.e.

$$T = \frac{\hbar'\kappa}{2\pi}, \tag{2.36}$$

$$S = \frac{\pi}{2\hbar}\left[\rho_0^2 + \rho_0\sqrt{\rho_0^2 - \alpha} - \alpha\ln\left(\rho_0 + \sqrt{\rho_0^2 - \alpha}\right)\right], \tag{2.37}$$

$$C = T\left(\frac{\partial S}{\partial T}\right)_{J,Q} = C_0 f\left[\frac{\hbar'}{\hbar}f - C_0\Delta\hbar\right]^{-1}, \tag{2.38}$$

where $\rho_0 = \sqrt{r_+^2 + a^2}$, and

$$\hbar' = \frac{2\hbar}{\alpha}\left(\rho_0^2 - \rho_0\sqrt{\rho_0^2 - \alpha}\right),$$

$$f = \pi\rho_0\sqrt{\rho_0^2 - \alpha},$$

$$C_0 = \frac{2\pi\rho_0^2(r_+ - M)}{3M + a^2/M - 2r_+}.$$

C_0 is the standard heat capacity of a Kerr-Newman black hole. If a Schwarzschild case is considered, the expressions (2.36)–(2.38) will reproduce the results of ref. [13].

It is easy to check that the expression (2.17) is still valid to a Kerr-Newman black hole, when $\rho_0^2 \gg \alpha$. The log-area type corrections have been obtained extensively in the literature [45]–[55] by several methods, such as conformal symmetry approach [45, 46], quantum geometry formalism [47–49] and string theory models [50, 51]. A prefactor of order unity is preferred in these models, although its precise value is in debate. Although some technical details are to be explored, it is not surprising that (2.17) agrees with the results obtained by string theory models and conformal symmetry approach, since GUP

originates from the literature on string theory and is related to the conformal invariance property of the fundamental string [4–6]. As to the relation between GUP and quantum geometry approach, it may be understood in an indirect manner. Concretely, we consider a modified dispersion relation (MDR), which has been suggested in the literature [56, 57] on loop quantum gravity. Due to the theoretical consistency, MDR should take the form as follows [18, 29]

$$E^2 = p^2 + m^2 + \theta l_p^2 E^4, \tag{2.39}$$

which leads to a log-area correction to entropy.[5] A similar relation has been suggested in a recently proposed and renormalizable theory of quantum gravity [59, 60], and a log term has been derived from the same theory too [61].

The link between GUP and MDR may be understood in the following way. On one hand, GUP changes the fundamental commutation relation of quantum theory; on the other hand, MDR originates from a quantized spacetime, where the quantum field theory should be different from that in a classical background. In other words, either GUP or MDR means a modification of the semiclassical theory, and therefore corrects the black hole thermodynamics. Certainly, the appropriate forms of GUP and MDR are crucial for the consistent correction to black hole entropy. For example, we consider a revision of generalized uncertainty principle [62]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha_1 l_p + \alpha_2 l_p^2 (\Delta p). \tag{2.40}$$

Following Bekenstein's analysis, we find that a $\sqrt{A}$ correction associated with the term linear in l_p can also be derived from (2.40). This is inconsistent with the result obtained by quantum geometry approach.

3 Black hole remnant

When we talk about a black hole remnant, we refer to a ground state that the thermal and non-thermal radiances vanish. This means that when a Kerr-Newman black hole decays to its ground state, it will lose all the initial charge and angular momentum, otherwise its energy can be extracted by Penrose process and superradiance [37, 44]. Therefore a Schwarzschild case was seriously considered in ref. [13]. In the semiclassical theory, a Schwarzschild black hole would evaporate to zero mass, and $C_0 \to 0$ as $M \to 0$. However, GUP results in two differences:

(i) the black hole acquires a nonzero minimum mass of order $\sqrt{\alpha}$;

(ii) the heat capacity vanishes as $M \to \sqrt{\alpha}/2$. This suggests the black hole's "zero point energy" be elevated to a higher scale. Therefore GUP may provide a mechanism to prevent a black hole from complete evaporation [13].

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[5]The term linear in l_p is absent in (2.39), otherwise it will lead to a $\sqrt{A}$ correction [18, 29]. Furthermore, this term may produce an unstable world [58].

A remnant can be derived from a dynamic black hole too. Considering a Vaidya black hole [63], its horizon is located by [64]–[66]

$$r_H = \frac{2m}{1 - 4\dot{m}},$$

where $\dot{m} = dm/dv$ is the mass loss rate. GUP restricts the black hole's radius by $r_H \geq \sqrt{\alpha}$, i.e.

$$\frac{2m}{1 - 4\dot{m}} \geq \sqrt{\alpha}. \tag{3.1}$$

For an evaporating black hole, its mass always decreases with time, i.e. $\dot{m} < 0$. Considering (3.1), we obtain

$$0 \leq -4\dot{m} \leq \frac{2m(v)}{\sqrt{\alpha}} - 1,$$

where $\dot{m} = 0$ denotes a black hole which stops evaporating. Obviously, $\dot{m} \to 0$ as $m(v) \to \sqrt{\alpha}/2$. Hawking radiation is shut off when the black hole evaporates to a Planck scale mass.

Black hole remnant has been suggested as an information loss reposition to resolve the black hole information problem [67, 68]. The remnant is assumed to retain the large information of the initial black hole although it has a small size and a tiny mass. However, this idea is questionable since it violates Bekenstein's entropy bound [69], $S \leq 2\pi E \ell / \hbar$, where E denotes the energy of the system of interest and ℓ the size. Following from this bound, the remnant's information content is a few bits at most. It is too tiny to resolve the information loss problem.

Can the situation be improved when a weaker constraint is considered? In an asymptotical de Sitter spacetime, the entropy of a matter system is restricted by the so-called D bound [70]

$$S_m \leq \frac{1}{4}(A_0 - A_c), \tag{3.2}$$

which is derived from the generalized second law via a Geroch process, where A_0 and A_c are the areas of the cosmological horizons of pure and asymptotical de Sitter spacetimes respectively. This consideration is motivated by the astronomical observation that the current universe is dominated by the dark energy. Cosmological constant Λ is the simplest candidate for the dark energy. The information capacity of a black hole remnant deserves to be seriously considered in the de Sitter spacetime. D-bound takes the form [70]

$$S_m \leq \pi r_g r_c, \tag{3.3}$$

when the gravitational radius of the matter system (r_g) is much less than the radius of the cosmological horizon (r_c). For a black hole remnant, its gravitational radius acquires the minimum value determined by GUP. Replacing $\sqrt{\alpha}$ for r_g, (3.3) is deduced to

$$S_r \leq \sqrt{\alpha}\pi r_c < \pi \sqrt{\frac{3\alpha}{\Lambda}}, \tag{3.4}$$

where we have considered $r_c < r_0 = \sqrt{3/\Lambda}$. Following from quantum statistical mechanics, the entropy bound (3.4) means that the number of the internal states of a black hole remnant is less than $\exp(\pi\sqrt{3\alpha/\Lambda})$. In other words, the information capacity of a black hole remnant in the de Sitter spacetime is restricted by the bound (3.4), which is concretely determined by the cosmological constant. In Planck units, the observed value of Λ is about 10^{-120}, and then S_r acquires the value of 10^{60} bits at most. D-bound allows a remnant to retain the large information, but the situation does not become optimistic. Considering a black hole of initial mass M_0, its entropy is $S_0 \approx 4\pi M_0^2$, which measures the total information hidden at the moment of collapse. For a solar mass black hole, its entropy is about 10^{76} bits, which is about 16 orders greater than S_r. This means that the remnant cannot retain the total information content of the initial black hole. The discrepancy becomes more serious when the larger black holes are considered. In order for the entropy bound (3.4) to be workable, the black hole mass must yield

$$M_0 < \left(\frac{3\alpha}{16\Lambda}\right)^{1/4} \sim 10^{30} m_p \sim 10^{25} g,$$

which is 8 orders less than the solar mass. Obviously, this mass scale would rule out most of the black holes in the universe. We therefore arrive at a conclusion that black hole remnants might not serve to resolve the information paradox.

Is it possible for the D-bound to be corrected by GUP? Since there are some similarities (in the sense of thermodynamics) between the cosmological horizon and black hole, we guess that a log type correction, $\ln(A_c/A_0)$, might appear in the r.h.s of (3.2). However, this correction is too tiny to overset the conclusion from the D-bound.

4   Summary

This research explores an alternative expression for black hole thermodynamics in the sense of GUP (1.1). We first consider a class of static and spherically symmetric black holes, and work out the expressions for the temperature, entropy and heat capacity. These quantities are expressed by (2.12), (2.16) and (2.19) respectively. The similar expressions are also valid to a Kerr-Newman black hole. For example, the temperature expressions, (2.12) and (2.36), can be expressed as a unified form ($T = \hbar'\kappa/2\pi$).

Our analysis is based on a gedanken experiment that a particle is absorbed by the black hole. The crucial problem is how to determine the characteristic size ρ_0 in the GUP for the black hole. In our opinion, ρ_0 must be a quantity associated with the event horizon, since the horizon is crucial for the black hole thermodynamics. For a class of static and spherically symmetric black holes, ρ_0 can be identified with r_0 ($\sim \sqrt{A}$) by intuition. Although dimensional analysis suggests another possibility: the inverse surface gravity κ^{-1}, $\sqrt{A}$ may be more consistent when a Kerr-Newman black hole is studied. Our preference is based on the following observations. Firstly, as showed by (2.30), there are two different axes by which the black hole surface is characterized as a confocal ellipsoid in rectangular coordinates. ρ_0 is indeed one of the axes and minimizes the change in horizon area. Secondly, the proportionality between the horizon area and the absorption

cross section means that ρ_0 is the characteristic size in the absorption process. Thirdly, ρ_0 represents the irreducible mass which is an invariant in a reversible process. This agrees with the thermodynamic requirement that the entropy is unchanged in a reversible process.

The proposal that $\rho_0 \sim \sqrt{A}$ may be valid to other stationary black holes. As an example, we consider a Kerr-Newman-(anti)de Sitter black hole, which is supposed to evolve into a Reissner-Nordström-(anti)de Sitter case by decreasing the angular momentum via a *reversible* process. The entropy of the latter is expressed as (2.17). Since the black hole's entropy and area are unchanged in a reversible process, the entropy of the Kerr-Newman-(anti)de Sitter black hole is also expressed as (2.17). This suggests that the characteristic sizes for both black holes should have a unified expression, i.e. $\rho_0 \sim \sqrt{A}$.

This proposal may suffer a modification, if we consider the possibility that the absorption cross section is corrected by an extended GUP as follows [17, 25–27, 71]

$$\Delta x \Delta p \geq 1 + \alpha(\Delta p)^2 + \beta(\Delta x)^2. \tag{4.1}$$

The β term represents the modification of gravity at large distances [17, 71], and $|\beta|^{-1/2}$ is usually interpreted as the (anti-)de Sitter radius.[6] Without loss of generality, the revised absorption cross section σ'_{abs} is a function of A, α and β, and it can be expanded as a series of α and β. Some additional corrections of order β^2 and α^2 (as well as the higher order terms) will be produced in the r.h.s. of (4.1), by identifying ρ_0 with $\sqrt{\sigma'_{abs}}$. Therefore the modification of the characteristic size is inessential, and $\rho_0 \sim \sqrt{A}$ is a good approximation for the black hole thermodynamics.

The extended GUP (4.1) has been utilized to discuss the black hole thermodynamics in the literature [17, 25–27], by identifying Δp with the temperature directly. A better understanding of (4.1) may be obtained by comparing the existing results with our argument. In particular, we hope to gain an insight into the link between (4.1) and cosmological constant problem. This subject is beyond the scope of the present paper, and it will be discussed in another work.

Acknowledgments

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